Outline

The St. Petersburg paradox

Time resolution

Standard resolution

Bernoulli's resolution

Menger's criticism

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The rôle of time in economics

Ole Peters
Imperial College London

ETH, Zurich, 01 March 2011

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Acknowledgments:

- A. Adamou, B. Meister, D. Farmer, Y. Schwarzkopf, F. Cooper, G. Pavliotis, J. Anderson, C. Collins, A. Rodriguez, M. Mauboussin, W. Miller, G. Soros, M. Gell-Mann, R. Hersh, D. Holm, O. Jenkinson, J. Lamb, J. Lawrence, C. McCarthy, J.-P. Onstwedder.

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- ZONlab ltd.

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Primer: Non-ergodicity

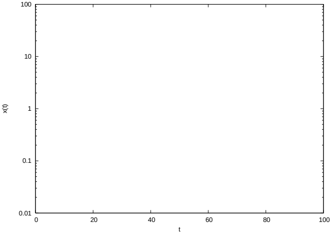
$dx = x(\mu dt + \sigma dW_t)$,
μ =drift term,
σ =volatility,
dW_t=Wiener process.

Simple model for dynamics of financial markets, biological populations, early stages of spreading epidemic...

Interesting statistical property: time-average growth rate.
Cannot assume that this is ensemble-average growth rate.
Let's try it out...

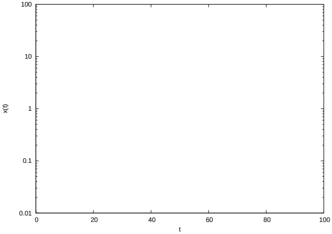
Choose parameters, $\mu = 0.05$, $\sigma = 0.45$.
Initial condition $x_0 = 1$.

<u>Ensemble average:</u>
Run N systems. Find ensemble average from large-N average.



Many systems.

<u>Time average:</u>
Run 1 system. Find time average from large-t behavior.



Long time.

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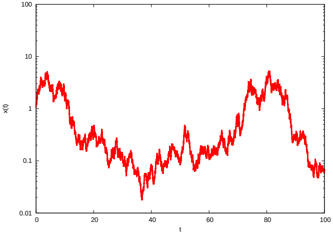

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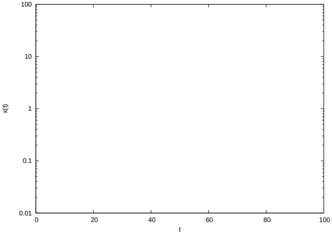

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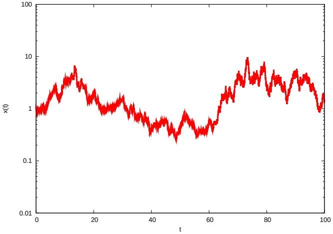
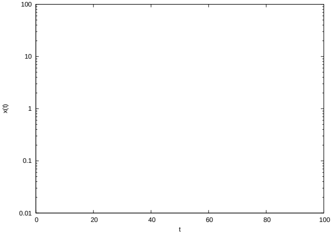


10 systems.

Time average:

Run 1 system. Find time average from large-t behavior.



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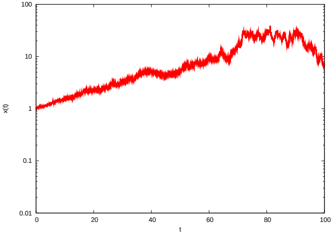

1,000 systems.

Time average:
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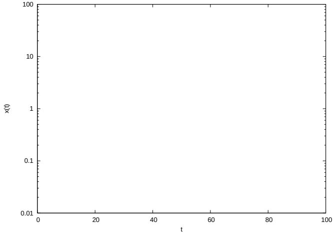

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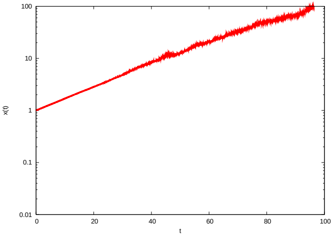

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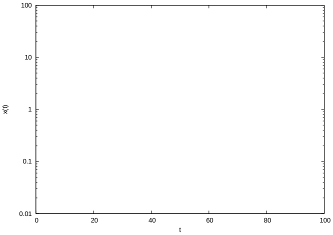

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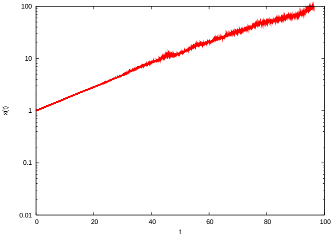

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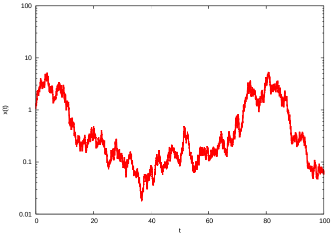

100 time steps.

Choose parameters, $\mu = 0.05$, $\sigma = 0.45$.
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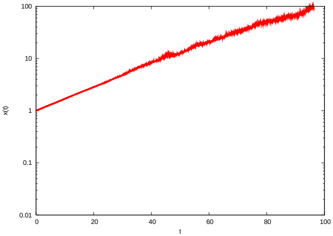

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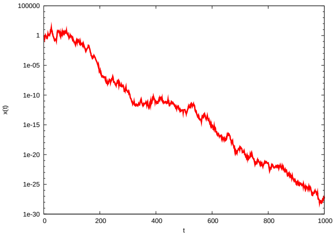

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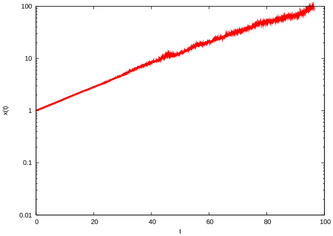

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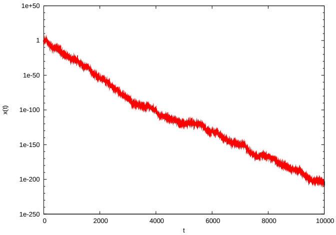

10,000 time steps.

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N. BERNOULLI to MONTMORT 1713:

402 *Extrait d'une Lettre, &c.*

c'eſt *A* qui joue, lequel en amenant un nombre pair prend un écu au jeu comme *B*; mais il ne met rien au jeu quand il amene un nombre impair, & ils continuent juſqu'à ce qu'il ne reſte plus rien au jeu, toujours avec cette condition, qu'ils prennent l'un & l'autre un écu du jeu quand ils amenent un nombre pair; mais que *B* ſeul met un écu au jeu quand il amene un nombre impair, on demande leurs ſorts. *Quatriéme Problême.* *A* promet de donner un écu à *B*, ſi avec un dé ordinaire il amene au premier coup ſix points, deux écus s'il amene le ſix au ſecond, trois écus s'il amene ce point au troiſiéme coup, quatre écus s'il l'amene au quatriéme, & ainſi de ſuite; on demande quelle eſt l'eſperance de *B*. *Cinquiéme Problême.* On demande la même choſe ſi *A* promet à *B* de lui donner des écus en cette progreſſion 1, 2, 4, 8, 16, &c. ou 1, 3, 9, 27, &c. ou 1, 4, 9, 16, 25, &c. ou 1, 8, 27, 64, &c. au lieu de 1, 2, 3, 4, 5, &c comme auparavant. Quoique ces Problêmes pour la plûpart ne ſoient pas difficiles, vous y trouverés pourtant quelque choſe de fort curieux : je vous ai déja propoſé le premier dans ma derniere Lettre. Vous me ferés plaiſir de me communiquer enfin votre ſolution du Her, afin que je puiſſe vous donner l'explication de mon Anagramme. Au reſte, Monſieur, je me réjouis de ce que votre ſanté eſt meilleure; mais je vous plains de ce que vous avés perdu votre Princeſſe. J'ai l'honneur d'être avec un attachement inviolable,

MONSIEUR,

Votre très humble & très obéiſſant Serviteur
N. BERNOULLY.

<u>Lottery:</u>
Toss a fair coin, find waiting time, n, to first *heads*.
Payout: 2^n.
$\rightarrow$ Expected payout: $\sum_n^\infty \left(\frac{1}{2}\right)^n 2^n = \sum_n^\infty 1$.

<u>"Paradox":</u>
No one wants to pay very much for a ticket.

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Why use expectation value as a criterion?

We're used to ergodicity:

expectation value = time-average in repeated games.

Strategy:
i) consider parallel rounds, for ensemble-average.
ii) consider sequential rounds, for time-average.

Result:
not equal, time average negative for small price, *i.e.* human behavior in line with time average.

Story:
– paradox introduced in 1713, long before ergodic theory.
– word "ergodicity" coined in 1882, concept clarified in 1931.
– today paradox can be resolved "physically".

Need to consider ticket price

$\longrightarrow$ compute growth factors per round $r = \frac{\text{net wealth after}}{\text{net wealth before}}$.

i) Ensemble average:

Growth factor in i^{th} realization: $r_i = \frac{w-c+m_i}{w}$

w – wealth before lottery.
c – cost of ticket.
m_i – payout in round i.

Finite-sample average: $\langle r \rangle_N = \frac{1}{N} \sum_i^N r_i$

Change summation to waiting times: $\langle r \rangle_N = \sum_n^{n_{\max}} \frac{k_n}{N} r_n$

k_n – frequency of waiting time n.
$n_{\max}$ – max waiting time observed in sample of N systems.

let N diverge
$\langle r \rangle := \lim_{N\to\infty} \langle r \rangle_N = \sum_n^\infty p_n r_n$

Ensemble-average growth factor (time unit = one round):

$$\langle r \rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n \frac{w-c+2^n}{w} = \left(1 - \frac{c}{w}\right) + \frac{1}{w} \underbrace{\sum_n^\infty \left(\frac{1}{2}\right)^n 2^n}_{\text{Bernoulli's divergence}}$$

$\rightarrow$ diverges positively for all c.

ii) Time average:

Need a dynamic: Assume multiplicative (*c.f.* savings account, population growth *etc*).

After T rounds, reach wealth $w \prod_i^T r_i$.

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Why? Cannot assume same starting wealth for all r_i.

Consider: $\prod_i^T r_i = \prod_{i=1}^{T'} r_i \prod_{j=T'+1}^{T} r_j$,
$\rightarrow$ large T: first product corresponds to equivalent games (*e.g.* same game, same wealth).

Can apply following arguments to first product.

ii) Time average:

Need a dynamic: Assume multiplicative (*c.f.* savings account, population growth *etc*).

After T rounds, reach wealth $w\prod_i^T r_i$.

Finite-time average: $\bar{r}_T = \left(\prod_i^T r_i\right)^{1/T}$

Change summation to waiting times: $\bar{r}_T = \left(\prod_n^{n_{\max}} r_n^{k_n}\right)^{1/T}$

let T diverge
$\bar{r} := \lim_{T\to\infty} \bar{r}_T = \prod_n^\infty r_n^{p_n}$

Time-average growth factor:
$\bar{r} = \prod_n^\infty r_n^{p_n}$

...

...

$\bar{r} = \prod_n^\infty r_n^{p_n} \rightarrow$ diverges? > 1? < 1?

Take logarithm:

$\ln(\bar{r}) = \ln\left(\prod_n^\infty r_n^{p_n}\right) = \sum_n^\infty p_n \ln(r_n) = \sum_n^\infty p_n \ln\left(\frac{w-c+2^n}{w}\right)$.

- Does not diverge.
- Meaning: time-average exponential growth rate (time unit = one round of lottery).
- Criterion for participation:
  $\bar{g} := \ln(\bar{r}) > 0$ — play.
  $\bar{g} < 0$ — don't play.

$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{\infty} \ln(r_i)$ looks like ensemble average. Why not?

Split time unit (1 round) into q sub-intervals.
Let r_j act for sub-interval, estimate
$\bar{g}_q^{\text{est}} = \sum_{j=1}^{q} (r_j^{1/q} - 1)$.

Take limit

$$
\begin{aligned}
\lim_{q\to\infty} \bar{g}_q^{\text{est}} &= \lim_{q\to\infty} \sum_{j=1}^{q} (r_j^{1/q} - 1) \\
&= \sum_{n=1}^{\infty} \lim_{q\to\infty} k_n (r_n^{1/q} - 1) \\
&= \sum_{n=1}^{\infty} p_n \lim_{q\to\infty} q (r_n^{1/q} - 1).
\end{aligned}
$$

Use definition of logarithm (inverse of exponential)
$\ln(r_i) := \lim_{q\to\infty} q (r_i^{1/q} - 1)$

Find $\sum_{k=1}^{\infty} p_n \lim_{q\to\infty} q(r_n^{1/q} - 1) = \sum_{k=1}^{\infty} p_n \ln(r_i)$

Message: logarithm implies time limit. Divergence of time, $\lim_{t\to\infty}$ avoided by re-scaling.

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Utility theory:

Monetary gains irrelevant – consider usefulness.
Utility: monotonically increasing, concave function of wealth (extra dollar worth more to poor person than to rich person).

D. BERNOULLI 1738: $u_B(w) = \ln(w)$

CRAMER 1728: $u_C(w) = \sqrt{w}$

MENGER 1934: $u_M(w) = \frac{Ww}{w+W}$

Evaluate gamble by computing expected gain in utility:

$\langle \Delta u \rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n u(w + 2^n - c) - u(w)$

Converges for u_B, u_C, u_M. Don't pay more than $c : \langle \Delta u \rangle = 0$

LAPLACE 1812: *"..whatever may be the function of the physical fortune which for each individual expresses his moral fortune."*

Bernoulli's function $u_B(w) = \ln(w)$:
$\langle \Delta u_B \rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n \ln(w + 2^n - c) - \ln(w)$

- Identical to time-average exponential growth rate $\langle \Delta u_B \rangle = \bar{g}(c, w) = \sum_n^\infty \left(\frac{1}{2}\right)^n \ln\left(\frac{w+2^n-c}{\ln(w)}\right)$
- Good behavioral guess by Bernoulli (evolution).
- Physical motivation (irreversibility of time, lack of ergodicity) not known to Bernoulli.
- No other utility function has similar physical interpretation.

Could have found solution earlier, ergodic theory from 1930s.

Another twist: erroneous paper by K. MENGER.

Motivation: BERNOULLI's resolution arbitrary, $u_B(w) = \ln(w)$ not justified physcially in 1738. MENGER 1934: *"ad hoc character"*, only *"apparent solution."*

→ find formal rejection. MENGER 1934: *".. solution .. according to .. logarithmic formula .. unsatisfactory on formal grounds."*

Idea: increase payouts fast enough with n to generate divergence in $\langle \Delta \ln(w) \rangle$.

- Problem: MENGER made a mistake.
- Big problem: No one noticed.
- Very big problem: rules out only physical solution.
- Result: time resolution rejected/peripheral in economics (KELLY, THORPE, COVER *etc.*).

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D. BERNOULLI's two-step argument: *"in a fair game the disutility to be suffered from losing must be equal to the utility to be derived from winning."*

i) Compute "utility to be derived from winning"
$\left\langle \Delta u_B^+ \right\rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n \ln(w + 2^n) - \ln(w)$

ii) Equate to "disutility to be suffered from losing"
$\left\langle \Delta u_B^+ \right\rangle - [\ln(w) - \ln(w - c)] = 0.$

Slightly different criterion than
$\langle \Delta u_B \rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n \ln(w + 2^n - c) - \ln(w) = 0.$
(try setting $w = c$.)
BERNOULL's motivation: impossible to pay more than w?

Note: arguments behavioral, no reason to insist on expected utility change.

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Change lottery payout, resurrect "paradox". For waiting time n receive $w\exp(2^n) - w$.

Follow Bernoulli's step i):
$\langle \Delta u_B^+ \rangle = \sum_n^\infty \left(\frac{1}{2}\right)^n \ln\left(\frac{w+\exp(2^n)-w}{w}\right) = \sum_n^\infty 1 \to$ diverges.

Behavioral: *"it is obvious that [..]no normal person would risk his total fortune or a substantial amount."*

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Why not? Worst case, $n = 1$, receive $w\exp(2) - w \approx 6w$. No problem paying $c = w$.

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Formal: cannot use unbounded utility functions.
Why not? Bernoulli's condition can be satisfied:

$$\sum_n^\infty \left(\frac{1}{2}\right)^n 2^n - [\ln(w) - \ln(w-c)] = 0.$$

As $c \to w$, positivity requires events with zero probability.

Note: Competing divergences, $\sum_n$ and $\ln(x \to 0)$. Insufficient to show that one diverges.

Menger 1934 forgot one or several of the following:

i) the lottery costs something.

ii) net loss is impossible unless $c > w(\exp(2) - 1)$.

iii) $\ln(x)$ diverges negatively as $x \to 0$.

iv) $\infty - \infty$ is not defined. Especially, $\infty - \infty \neq \infty$.

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ARROW 2009: *"..a deeper understanding was achieved only with* KARL MENGER*'s paper (1934)."*

Recommendations for MENGER's game:

- Time resolution (like expected logarithmic utility change): play, provided bankruptcy impossible.
  - reasonable if game played in sequence many times.
  - probably no practical significance because game extreme.
- Bernoulli's resolution: play, provided $c < w$.
  - arbitrary: minimum net gain $= w\exp(2) - 2w \approx 5.34w$.

Aside: MENGER's game typically cited as payout $\exp(2^n)$, not original game with $w\exp(2^n) - w$.

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- 1654: PASCAL and FERMAT introduce expectation value.
- 1713: N. BERNOULLI notices absurd recommendation.
- 1728/1738: D. BERNOULLI and CRAMER introduce utility.
- 1872/1879: BOLTZMANN and MAXWELL worry about ergodicity.
- 1931: BIRKHOFF speficies conditions for ergodicity $\rightarrow$ N. BERNOULLI's problem now solvable.
- 1934: MENGER's paper rules out log-utility and by implication time resolution.

Result: large part of economics (utility theory, game theory, welfare economics, risk management, behavioral economics...) largely misses time argument.

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What if we use time argument?

The time resolution of the St. Petersburg paradox. OP, Phil. Trans. R. Soc. A (2011), in press.

- Can optimize leverage, find objective limits for risk-taking.

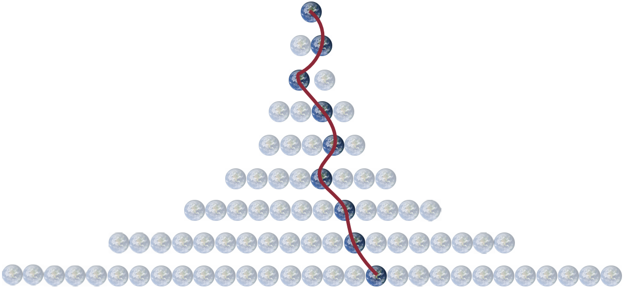
  Optimal leverage from non-ergodicity. OP, Quant. Fin. (2010) doi:10.1007/s10955-010-0039-0.

- Can use higher-order efficiency arguments (no leveraged-arbitrage), predict stochastic properties of stock markets, agree with observations.

  Stochastic market efficiency. OP and A. Adamou, arXiv:1101.4548 (2011).

- Can use this to assess market stability, *e.g.* housing.
- Can find better measures of economic prosperity (work in progress).
- Some behavior appears "irrational" from ensemble-average perspective. Wrong perspective because system not ergodic. May be rational from time perspective. *E.g.* risk aversion.
- Can derive "reasonable" behavior through evolution (time).

Help!



Thank you.