Mathematical Models of Traffic Flow: Macroscopic and Microscopic Aspects
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Introduction

Broad subject! So many aspects of traffic modeling, e.g.

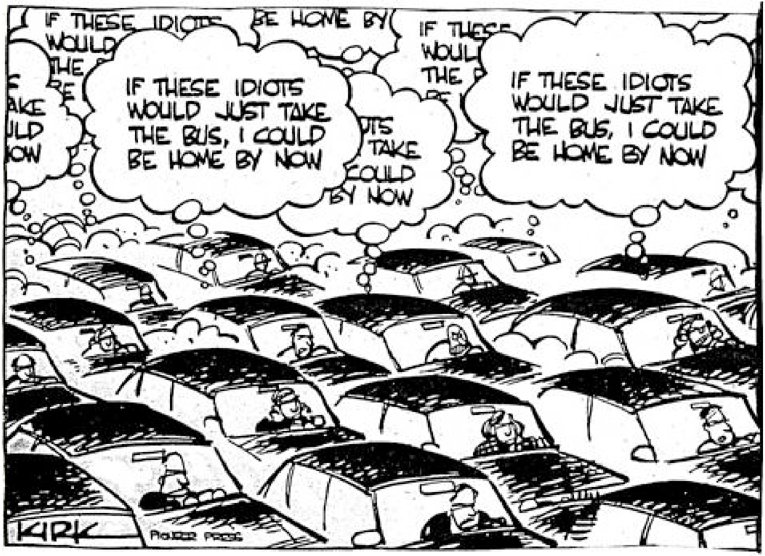
- Fully (cellular automata, numerical schemes ...) or semi-discrete (ODE, delayed ODE ...) / Macroscopic (PDE (hyperbolic (conservation laws? Hamilton-Jacobi? With diffusion and/or relaxation?)
- Or Mesoscopic (kinetic description)?
- Multiscale (structure of traffic jams, "phase transitions", homogenization, hybrid schemes ...)
- (I): Instability, e.g. stop and go waves / (S): Stability: preserve nonnegative speed (!) and (hopefully!) no crash ...
- ODE description much better for (I) and PDE for (S) ... How to find the right combination? Related question: if necessary, give priority to ODE and use "modified equation at higher order" for describing specific effects?
- Junctions, link with homogenization. Networks. Hybrid schemes ... I won't cover everything!

Outline

- Introduction
- Discrete / Fluid Models
- The Fluid Model (Without Relaxation)
  - The Eulerian System
  - Riemann Problem. Waves
  - Motivations. Lagrangian version
  - Link with Microscopic Models (FLM)
  - Lagrangian Godunov Scheme
  - *Passing to the limit(s)*
- Junctions
  - On a network
  - Ingoing Half-Riemann Problem
  - Outgoing Half-Riemann Problem
  - Riemann Problem at a junction
  - 2-1 Junction: Homogenization
  - Homogenized Supply
  - Conclusion on junctions

Outline ...

- With Relaxation: Traveling Waves and Oscillations
  - Motivations
  - Remark: Whitham Subcharacteristic condition
  - Smooth "simple waves" are generically Traveling Waves
  - J. Greenberg's work periodic solutions for ARG. Extensions...
  - An Example: the Intelligent Driver Model
  - Additional Remarks. Conclusion
- Comments and references


IF THESE IDIOTS WOULD JUST TAKE THE BUS, I COULD BE HOME BY NOW
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KIRK
PIONEER PRESS

The German car industry trying to catch up with its French competitors (allegory).

Discrete / Fluid Models

- (Fully or) 1/2 discrete: Follow the Leader Models...
  Car length: $l = \Delta X$. Spacing:
  $\tau_j := x_{j+1} - x_j\,;\, s_j = 1/\rho_j \ = \tau_j/l$
  specific volume, density.

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{s}_j = \frac{v_{j+1}-v_j}{l} \\ \dot{v}_j = F(x_j, x_{j+1}, v_j, v_{j+1}) \\ (e.g.) = \alpha v_j^m \, V'(\frac{x_{j+1}-x_j}{l})\,\frac{v_{j+1}-v_j}{l} + \beta(V_e(\frac{x_{j+1}-x_j}{l}) - v_j) \end{cases} \tag{2.1}$$

Convective part (fast reaction) + (slow) relaxation part ...
Examples, see also Gazis-Herman-Rothery and ...

- $\alpha = 0, \beta > 0$: Bando's Optimal Velocity Model
- $\alpha > 0, \beta = m = 0$: Aw-Klar-Materne-Rascle, SIAP 2002
- $\alpha > 0, \beta > 0, m = 0$: J. Greenberg and/or Aw-Rascle, SIAP 2000-2004
- Intelligent Driver Model (IDM): Helbing-Treiber, ∼ 2000
  $\dot{v}_j = a\,[1 - v_j^m - (\frac{s_b(v_j) - v_j(v_{j+1}-v_j)}{s_j})^2]\,;\, s_b(v) := s_0 + s_1\sqrt{v} + s_2(v)$

- Kinetic:

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- Fluid:
  - First Order: Lighthill-Whitham-Richards (LWR) [↔ Hamilton-Jacobi]

$$\partial_t \rho + \partial_x(\rho v) = 0,\ v = V(\rho), V'(\rho) < 0, (\rho V)" < 0,$$

Fundamental diagram: flux $q = \rho V(\rho)$.
Riemann Pb: $\rho(x,0) = \rho_\pm$ for $\pm x > 0$:
- centered rarefaction waves (acceleration) if $v_- < v_+$,
- shock waves (braking) if $v_- > v_+$. Very robust, (too) stable. Figures.

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      - shock waves (braking) if $v_- > v_+$. Very robust, (too) stable. Figures.
    - Second Order: Payne-Whitham (cf Gas Dynamics)

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + v\, \partial_x v = -\rho^{-1} p'(\rho) \partial_x \rho + \ldots := -\tilde{p}'(\rho) \partial_x \rho + \ldots \end{cases}$$

    - Daganzo (Requiem, 95) PW is a terrible model!! [Diffusion still worse !]
      Paradoxes: 1: $v < 0$ **and 2:** $\lambda_2 = v + c > v$ **!!**
    - Aw-Rascle (Resurrection ?, 2000), Zhang(2002). Fixing:
      $\partial_x p \to \partial_t p + v \partial_x p$
    - Second equation in (PW) becomes:

$$\partial_t v + v\, \partial_x v = -\tilde{p}'(\rho)(\partial_t + v \partial_x)(\rho)$$

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   - The Eulerian System
   - Motivations. Lagrangian version
   - Link with microscopic models (FLM)
   - Lagrangian Godunov Scheme
   - Application: hybrid Lagrangian schemes
   - Passing to the limit(s)
3. Junctions
   - On a network
   - Ingoing Half-Riemann Problem
   - Outgoing Half Riemann Problem
   - Riemann Problem at a junction: Principle ...
   - 2-1 Junction: Homogenization
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   - Remark: Whitham Subcharacteristic Condition
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The Fluid Model. Eulerian System

- Therefore, setting (new) $p(\rho) := \tilde{p}(\rho)$,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t w + v\, \partial_x w = 0. \end{cases} \tag{3.1}$$

  Here, w : Lagrangian marker ("color") **defines the fundamental diagram**, e.g. $w := v + p(\rho) := v + v_{max} - V(\rho)$ or (better) $w := v - V(\rho)$, could be much more general (aggressivity, origin, destination, alive (?) for pedestrians, size of a file ...)

- In conservative form, *the system becomes (E):*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0 \end{cases} \tag{3.2}$$

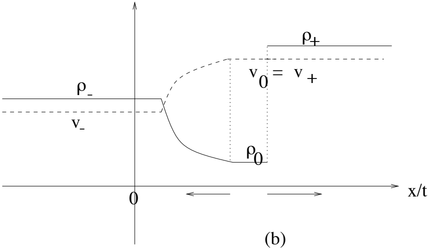
- *Here $V(\rho)$ is a known function, with $V'(\rho) < 0$ and (strict concavity, again can be extended !), λ_1 is GNL: either shocks or rarefactions*

Riemann Problem (RP). (Very) quick version

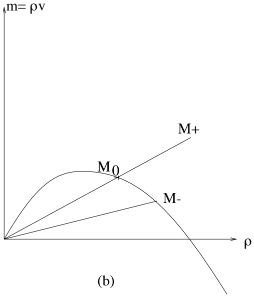
- **Riemann Problem: IVP with** $U(x,0) = U_\pm$ **for** $\pm x > 0$
- Strictly hyperbolic system, (except for $\rho = 0$ ...)
- Eigenvalues of 2x2 matrix : $\lambda_1(U) = v + \rho V'(\rho) < \lambda_2(U) = v$
- λ_1 : genuinely nonlinear rarefaction (acceleration) or shock (braking), whose curves coincide here, since (Rankine-Hugoniot) $[\rho(v-\sigma)] = 0$ and $[\rho w(v-\sigma)] = ((\rho(v-\sigma)_\pm) \cdot [w] = 0.$ Q: Why?
- λ_2 *is linearly degenerate : 2-contact discontinuity.*
- *Diagonalization: Riemann invariants (say on road i) :*

$$\begin{gathered} w(U) := w_i(U) = v - V_i(\rho) \text{ and } v(U) = v \\ \partial_t w + v\,\partial_x w = 0, \quad \partial_t v + \lambda_{1,i}(U)\partial_x v \approx 0 \end{gathered}$$

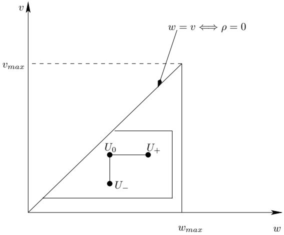
Figures

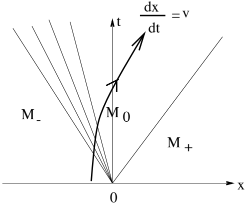

ρ+
v0 = v+
ρ-
v-
ρ0
0
x/t

(b)


m= ρv
M+
M0
M-
ρ

(b)


v
$w = v \iff \rho = 0$
v_{max}
U_0
U_+
U_-
w_{max}
w


t
$\frac{dx}{dt} = v$
M-
M0
M+
0
x

(b)

- **Solution of Riemann Pb** *with initial data* U_- *and* U_+*:* **first find** U_0 **with same** w **as** U_- **and same** v **as** U_+**.** *Then, see Figure, construct:*
  - *a* $1-$ *wave connecting* U_- *and* U_0 *by a shock or rarefaction as for first order model, with fundamental diagram* $v = V(\rho) + w(U_-)$ *, followed with vacuum state if* $v_- < v_{max}(w_-)vv_+$*, see Remark below :*
    - *a rarefaction:* $w(U_0) := v_0 - V(\rho_0) = w(U_-)$*, if* $v_0 = v_+ > v_-$*,*
    - *or a shock:* $w(U_0) := v_0 - V(\rho_0) = w(U_-)$*, if* $v_+ > v_-$ *(coinciding),*
  - *followed by a* $2-$ *wave between* U_0 *and* U_+*: contact discontinuity:* $v_0 = v_+$
- **In all cases***,if* $d(U^1, U^2)) := |v_1 - v_2| + |w_1 - w_2|$*, then (BV estimates) (no wild oscillation)*

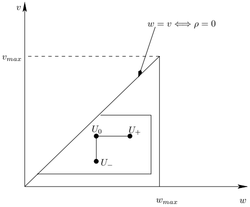
$$d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+),$$

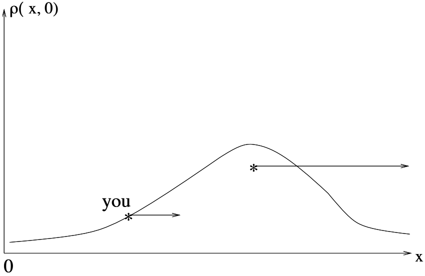
and **(bounded) rectangles in** (v, w) **plane are invariant regions:** L^∞ **estimates. No more paradox 1** *(*$v < 0$*) or 2 (*$\lambda_2 > v \geq 0$*). No crash if no crazy driver (again, invariant region) ... Compare with PW, or Bando!*

Remarks

- Coinciding curves, since the "color" does not change when braking!
- Exercise 1: in the general case: $v = V(\rho, w)$ with $V(., w)$ strictly decreasing, show that $\lambda_1 = V + \rho \frac{\partial V(.,w)}{\partial \rho}$, i.e. λ_1 is the slope of the tangent to the curve: $\rho \mapsto \rho.v$ in the $(\rho, \rho.v)$ plane. Compare to Rankine-Hugoniot for first equation ...
- Exercise 2: ... and that λ_1 is GNL iff this curve is either strictly concave or strictly convex. Moreover, in the first case, show that braking corresponds to a shock and conversely: instantaneous braking.
- Under these assumptions, show that **vacuum** appears in Riemann Problem iff $v_- < v_{max}(w_-) < v_+$. In this case, we define "vacuum" (although there can be many cars ahead ...) as the region $\{tv_{max}(w_-) < x < tv_+\}$. Note that this region is not accessible to the cars U_-.

Riemann Pb in (v, w) plane here with $w = v + p(\rho) = v + v_{max} - V(\rho)$. BV estimate: $d(U^-, U^+) = d(U^-, U^0) + d(U^0, U^+)$. No oscillation ...



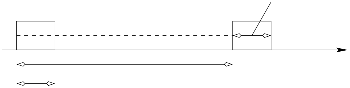


First motivation: x **or** t **dependence?** Do we react to flow variations in x or t : if the "wave" is faster than you, should you brake (cf gas dynamics), or accelerate (cf our model)?? Compare:

$\partial_t v + v\,\partial_x v = -\,\partial_x\,\tilde{p}(\rho)$ or $= -(\partial_t + v\,\partial_x)(\tilde{p}(\rho))$

Motivations. Lagrangian version

- Lagrangian mass coordinates (Courant-Friedrichs)



- From mass conservation,

$$\partial_t \partial_x X = \partial_x \partial_t X \,; \quad X(x,t) = \int_{-\infty}^{x} \rho(y,t)dy$$

  Essentially, $X = -N$, N : cumulated flow. Discrete X_j = position of car j if parked nose to tail. Also, s is additive (on a single lane), not ρ !! Important for homogenization.

- *s and ρ are adimensional (occupancy), therefore invariant in a* **hyperbolic scaling:** *let a zoom parameter $\varepsilon \to 0$ and* $(x', t', X', \Delta t', \Delta X') := \varepsilon\,(x, t, X, \Delta t, \Delta X)$

Remarks. Exercises:

- Show details of the change of variables: $\{(x, t) \mapsto (X, T := t)\}$. Compute the partial derivatives in (x, t) in terms of those in (X, T) and conversely.
- Show that the mass conservation in Eulerian system (E) implies the first equation of (3.5) below (conservation of space).
- What happens in the above change of variable when vacuum occurs?
- Show that the two systems (E) and (3.5) have the same strict Riemann invariants v and w, and that a characteristic speed λ_E for (E) corresponds to a characteristic speed λ_L for (3.5), with $\lambda_E = v + \rho.\lambda_L$. Solve the Riemann Problem for (L).
- Show that in the general case $v = V_1(\rho, w) = V(s = 1/\rho, w)$, for any entropy-flux pair $(\eta(s, w), q(s, w))$ for (3.5), i.e. for any additional conservation law of the form:

  $$\partial_t \eta + \partial_X q = 0,$$

  satisfied by *any* smooth solution of (3.5) q must be an arbitrary smooth scalar function of $v = V(s, w)$. If $\{s \to V(s, w)\}$ is increasing and concave, check that the entropy η is convex in s iff q is concave in v.

Link with microscopic Models (FLM)

- Follow The Leader Model (FLM) . We set $w = v - V(\rho)$ or $v - V(s)$

$$\begin{cases} \dot{x}_j = v_j \Longrightarrow \dot{s}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{v}_j = V'\left(\frac{x_j+1-x_j}{\Delta X}\right)\frac{v_j+1-v_j}{\Delta X} = V'(s_j)\dot{s}_j \end{cases} \tag{3.3}$$

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- Equivalent form (FLM'):

$$\begin{cases} \dot{s}_j = \frac{v_j+1-v_j}{\Delta X} \\ \\ \dot{w}_j = 0 \quad ; \quad w_j := v_j - V(s_j) \end{cases} \tag{3.4}$$

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- When $\Delta X \to 0$, (FLM') formally (in fact, rigorously) CV to Lagrangian System (L) (which is $\Leftrightarrow$ Eulerian System (E)):

$$\begin{cases} \partial_t s - \partial_X v = 0\,,\ s := \rho^{-1}\,, \\ \partial_t w = 0\,,\ w = v - V(s) := v - \tilde{V}(\rho). \end{cases} \tag{3.5}$$

- Now the first order Euler explicit discretization of (FLM'):

$$\begin{cases} s_j^{n+1} = s_j^n + \frac{\Delta t}{\Delta X}\left(v_{j+1}^n - v_j^n\right) \\ w_j^{n+1} = w_j^n = ... = w_j \; ... \end{cases} \tag{3.6}$$

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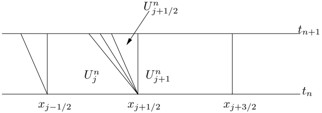
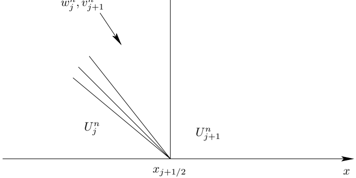
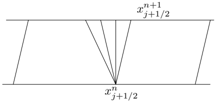
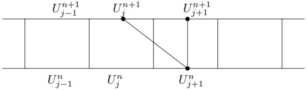
- is **exactly** the Godunov approximation of Lagrangian system and (**exceptional**) has the same stability as the Riemann Pb (in each Lagrangian cell, v is monotonous, since $w = C$).
- Therefore, when $\Delta t \to 0$, with ΔX fixed, (3.6) CV to (FLM') **which inherits the same invariant regions and BV-stability properties** (not obvious directly!).
- *Even for weak solutions (Wagner, 87) (L) is equivalent to system (E).*
- *Eigenvalues become:* $\lambda_1 = -V'(s) < 0$ *(GNL), and* $\lambda_2 = 0$ *(LD), with same Riemann Invariants* v, w *and same structure (coinciding ...)*

Lagrangian Godunov Scheme









In Eulerian moving coordinates, $x^{n+1}_{j+1/2} = x^n_{j+1/2} + \Delta t \, v^n_{j+1}$. Therefore

$$\begin{cases} s^{n+1}_j = \frac{x^{n+1}_{j+1/2} - x^{n+1}_{j-1/2}}{\Delta X} = s^n_j + \frac{\Delta t}{\Delta X} \, (v^n_{j+1} - v^n_j), \\ w^{n+1}_j = w^n_j = ... = w_j \end{cases} \tag{3.7}$$

the Godunov scheme for (3.5) is **exactly** (3.6) and defines numerically the trajectories. Finally, as w remains constant in each cell, by monotonicity, the new $v = v^{n+1}_j = w^{n+1}_j + V(s^{n+1}_j)$ is between v^n_{j+1} and v^{n+1}_j.

Case with relaxation: Fractional Step

- Prototype:

$$\begin{cases} \partial_t s - \partial_X v = 0\,,\ s := \rho^{-1}\,, \\ \partial_t w = V_e(s) - v\,,\ w = v - V(s) := v - \tilde{V}(\rho). \end{cases} \tag{3.8}$$

- First half step: as above , now called $U^{n+1/2}$:

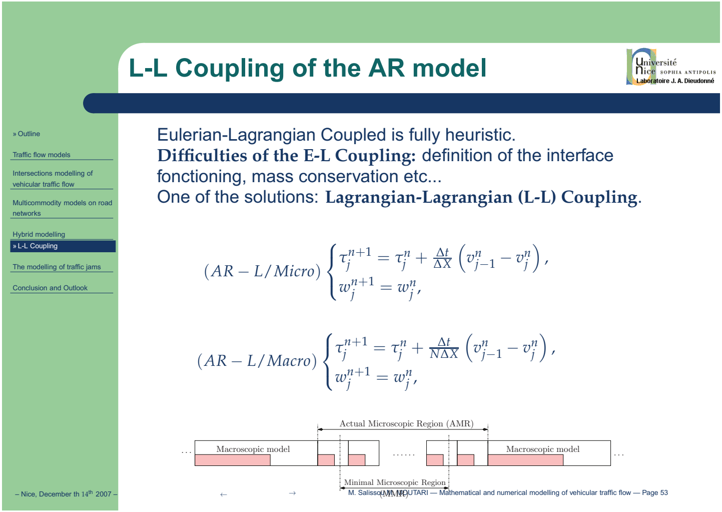
$$\begin{cases} s_j^{n+1/2} = s_j^n + \Delta t\ \frac{v_{j+1}^n - v_j^n}{\Delta X}, \\ w_j^{n+1/2} = w_j^n\,,\ v_j^{n+1/2} = w_j^n + V(s_j^{n+1/2}). \end{cases} \tag{3.9}$$

- Second half-step: $s_j^{n+1/2} = s_j^{n+1/2}$. Approximate the ODE (3.8,ii):

$$\begin{cases} w_j^{n+1} = e^{-\Delta t}.w_j^{n+1/2} + (1 - e^{-\Delta t}).V_e(s_j^{n+1/2}), \\ v_j^{n+1} = w_j^{n+1} + V(s_j^{n+1/2}). \end{cases} \tag{3.10}$$

Example of application: hybrid Lagrangian schemes

Here without relaxation, joint work with S. Moutari (SIAP 2008)



L-L Coupling of the AR model

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Traffic flow models
Intersections modelling of vehicular traffic flow
Multicommodity models on road networks
Hybrid modelling
» L-L Coupling
The modelling of traffic jams
Conclusion and Outlook

Eulerian-Lagrangian Coupled is fully heuristic.
Difficulties of the E-L Coupling: definition of the interface fonctioning, mass conservation etc...
One of the solutions: **Lagrangian-Lagrangian (L-L) Coupling**.

$$(AR - L/Micro) \begin{cases} \tau_j^{n+1} = \tau_j^n + \frac{\Delta t}{\Delta X}\left(v_{j-1}^n - v_j^n\right), \\ w_j^{n+1} = w_j^n, \end{cases}$$

$$(AR - L/Macro) \begin{cases} \tau_j^{n+1} = \tau_j^n + \frac{\Delta t}{N\Delta X}\left(v_{j-1}^n - v_j^n\right), \\ w_j^{n+1} = w_j^n, \end{cases}$$

– Nice, December th 14th 2007 –

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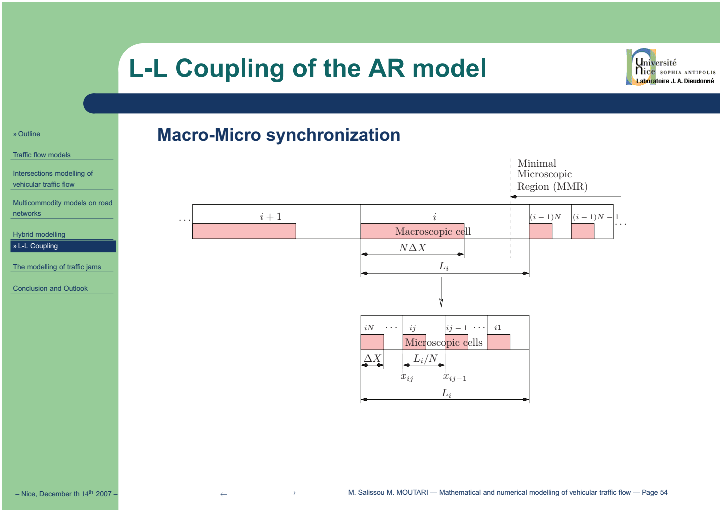
Lagrangian interfaces, **minimal fixed** Eulerian microscopic region



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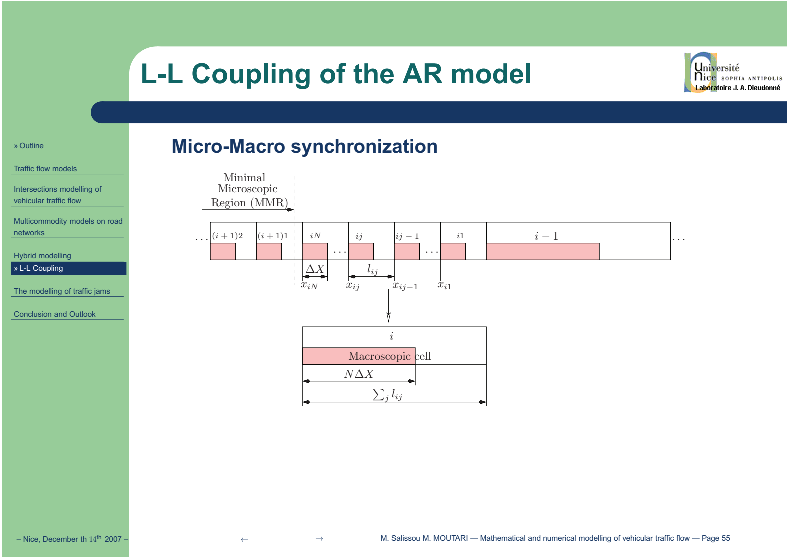

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Micro-Macro synchronization

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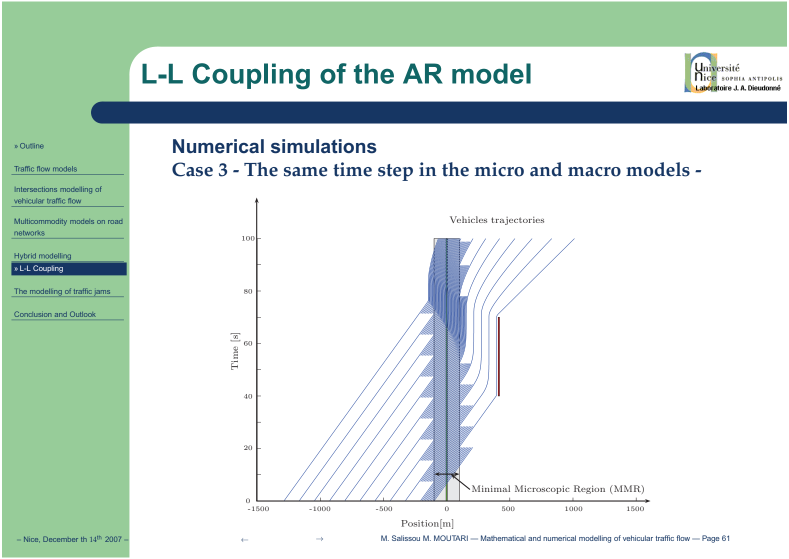

L-L Coupling of the AR model

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- » Outline
- Traffic flow models
- Intersections modelling of vehicular traffic flow
- Multicommodity models on road networks
- Hybrid modelling
- » L-L Coupling
- The modelling of traffic jams
- Conclusion and Outlook

Numerical simulations

Case 3 - The same time step in the micro and macro models -

– Nice, December th 14th 2007 –

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Passing to the limit(s) (without relaxation)

- Thanks to uniform BV estimates **and invariant regions**, we can either let $\Delta t \to 0$, with ΔX fixed: (GOD) $\equiv$ the explicit Euler scheme CV to (FLM), with again same BV and L^∞ estimates (again, not obvious directly!) Finally and then (FLM) $\equiv$ the semi-discretization CV to (L), when $\Delta X \to 0$.
- **Exercise:** *check details: BV estimates à la Glimm, convergence, uniqueness: "Krushkov":* $\eta_k = \mathrm{sgn}(v-k).(S(v,w) - S(k,w))$; $q_k(v) = -|v-k|$.

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- **Exercise:** *check details: BV estimates à la Glimm, convergence, uniqueness: "Krushkov":* $\eta_k = \mathrm{sgn}(v - k).(S(v,w) - S(k,w))$; $q_k(v) = -|v - k|$.
- or start again from (GOD), then let ΔX and $\Delta t \to 0$ together with a fixed ratio and CFL condition: then (GOD) CV to the same unique solution to (L): commutation of limits.
- By-product: (FLM) CV to (L): that can be used directly, avoiding scaling pbs (especially with a relaxation term), cf Colombo, Marcellini, Rascle, to appear SIAP. Similar results in singular limit works with Berthelin, Degond, Delitala *et al* ...

- With no relaxation term, this procedure combines nicely with a hyperbolic scaling, with a zoom parameter $\epsilon \to 0$: and $(x', t', X', \Delta t', \Delta X') := \epsilon (x, t, X, \Delta t, \Delta X)$.
- *ρ, s, v, system (L) and (God) are unchanged in this scaling, but not the* **initial data**

$$U_0(X, \epsilon X) := U_0(\frac{X'}{\epsilon}, X')$$

- *Therefore,* **if there is no small scale** $\frac{X'}{\epsilon}$ *in the initial data the solution of (God) converges to the* **(unique)** *solution of (L) when* $\epsilon \to 0$*: with Aw-Klar-Materne-Rascle, SIAP 2002)*
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- *First ∃ result (no scaling): J. Greenberg (SIAP 2001), with Relax, (sub)"characteristic" case; Aw, PhD*
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- *Oscillations in w (mixture) on outgoing roads in junctions: with Herty, Moutari, see further*

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- *Oscillations in w (mixture) on outgoing roads in junctions: with Herty, Moutari, see further*
- *Summary: start from (FLM'), make $\rho = \rho_j(t)$ (constant in space) (Eulerian) or $s = s_j(t)$ (Lagrangian) between two cars $j, j+1$ ...*

On a network. Cauchy Problem

- We do not specify here the relations with exterior world ...
- Conservative form on each road, with the same choice: $y_i = \rho_i w_i = \rho_i(v_i - V_i(\rho_i))$.:

$$\partial_t \begin{pmatrix} \rho_i \\ y_i \end{pmatrix} + \partial_x \begin{pmatrix} y_i + \rho_i V_i(\rho_i) \\ y_i(y_i + \rho_i V_i(\rho_i))/\rho_i \end{pmatrix} = 0,$$

  with the previous choice: $y_i = \rho_i w_i = \rho_i(v_i - V_i(\rho_i))$.
- Rankine-Hugoniot conditions through a junction, with $\{b_i, i \in \delta_-\}$ (incoming roads) and $\{a_j, j \in \delta_+\}$ (outgoing roads):

$$\sum_{i\in\delta^-} (\rho_i v_i)(b_i^-, t) = \sum_{j\in\delta^+} (\rho_j v_j)(a_j^+, t)$$

$$\sum_{i\in\delta^-} (\rho_i v_i w_i)(b_i^-, t) = \sum_{j\in\delta^+} (\rho_j v_j w_j)(a_j^+, t)$$

In other words, a weak (entropy) solution on a network must :

- be a weak (entropy) solution on each road i
- conserve the total number of cars and also the total number of cars of each "color" w, at all junctions, where
- $\forall i \in \delta^-$: incoming, and for all $j \in \delta^-$: outgoing road, the unknown limit values U_i^+ at $b_i - 0$ and U_j^- at $a_j + 0$ (Attention !!), have to be determined below.
- At an arbitrary junction, we want to solve the Riemann Problem, i.e. the Initial Value Problem, by solve a half- Riemann Problem on each road, in which the speed of all (centered) waves is constrained to be ≤ 0 on ingoing roads and ≥ 0 on outgoing roads.

Ingoing Half-Riemann Problem

- Consider an ingoing road, and assume that we knwow its actual outgoing flux q at the junction.
- Then we would like to connect the left Riemann data $U_i^- = (\rho_i^-, v_i^-)$ through a 1-wave of nonpositive speed to a state

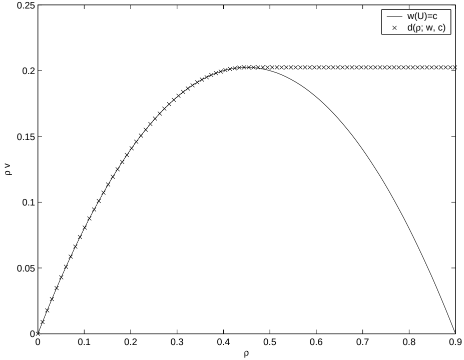
$$U_i^+ = \{w_i(U) := v_i - V_i(\rho) = w_i(U_i^-)\} \cap \{\rho v = q\}$$

- This is not always possible, see Figure below, and moreover we would like q to be as large as possible.
- **Definition** (Lebacque): *The* **demand** $d(U_i^-) :=$ *is the maximal possible flux* $q = \rho v$*, for any* U *connected to* U_i^- *by waves of nonpositive speed (necessarily* $1-$*waves) and satisfying*

$$w_i(U) := v_i - V_i(\rho) = w_i(U_i^-).$$

- We recall that eventually the actual $U_i^+ = U_i(b_i - 0, t)$ (in) and $U_j^- = U_j(a_j + 0, t)$ (out) must satisfy Rankine-Hugoniot.

Demand. Figure



Graph of the demand $d(U)$: in uncongested regime, i.e. if $\rho_i^- < \tilde{\rho}$ (the sonic point), then the maximal flux $\tilde{q}$ at a point U on curve $w(U) = w(U_i^-)$ which can be connected with U_i^- by a wave of speed ≤ 0 is reached for $U = U_i^-$ itself (and the other point on this curve with same flux). Conversely, if $\rho_i^- > \tilde{\rho}$, then $\tilde{q}$ corresponds to $\rho = \tilde{\rho}$, i.e. to the maximal possible flux on this fundamental diagram: $w_i(U) = w_i(U_i^-)$.

Outgoing Half Riemann Problem

- On the outgoing axes, we want to connect the right Riemann data $U_j^+ = (\rho_j^+, v_j^+)$ to a state U_j^- by waves of nonnegative speed(s).
- First connect $U_j^+ = (\rho_j^+, v_j^+)$ through a 2-contact discontinuity (of speed $v_j^+ > 0$) to a first intermediate state U_j^*, still unknown
- Here, U_j^* comes from road i, but is on road j. Therefore $w_j(U_j^*) := w_j^* := w_i(U_i^-)$!!.
- If, e.g. $w_j(U_j^*) := w_j^* = w_i(U_i^-) = 20$ km/h, this (same) driver will drive 20 km/h faster than $V_i(\rho)$ on road i and than $V_j(\rho)$ on road j: whatever the road conditions are, he likes to drive 20 km/h faster than the local $V(\rho)$. So

$$U_j^* = \{w_j(U) := v - V_j(\rho) = w_j^*\} \cap \{v = v_j^+\}$$

- Now, as for the demand, we define the supply associated with the state U_j^* and the above fundamental diagram: $w_j(U_j^*) = w_i(U_i^-)$.

Supply

- Again, assume we know the actual ingoing flux at the junction on this outgoing road j. Then, having already connected U_j^+ to U_j^* by a $2-$wave (of speed ≥ 0), we would like to connect U_j^*, through a 1-wave of nonnegative speed, to a state $U = U_j^-$, still unknown, which will play near $x = 0^+$ the same role as U_i^+ near $x = 0^-$ (Attention !). Therefore, if this is possible,

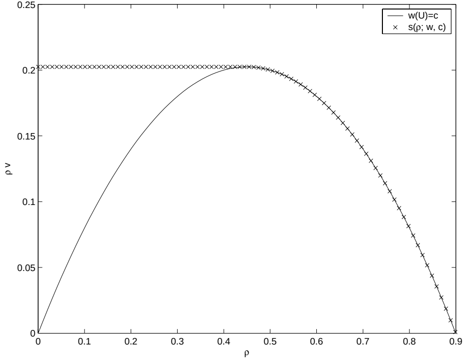
$$U_j^- = \{w_j(U) = w_j^*\} \cap \{\rho_j v = q\},$$

with hopefully q as large as possible.

- **Definition** (Lebacque): *The* **supply** $s(U_j^*)$ *is the maximal possible flux* $q = \rho v$, *for any* $U = U_-^j$ *connected to* U_j^* *by waves of nonnegative speed and satisfying*

$$w_j(U) = w_j^* = w_i(U_i^-).$$

Supply: Figure



Graph of the supply $s(U)$: in uncongested regime, i.e. if $\rho_i^- < \tilde{\rho}$ (the sonic point), then the maximal flux $\tilde{q}$ at a point U on curve $w(U) = w_j(U_j^*)$ which can be connected with U_j^* by a wave of speed ≥ 0 is reached for $U = U_j^*$ itself (and the other point on this curve with same flux). Conversely, if $\rho_i^- > \tilde{\rho}$, then $\tilde{q}$ corresponds to $\rho = \tilde{\rho}$, i.e. to the maximal possible flux on this fundamental diagram: $w_j(U) = w_j(U_j^*) = w_i(U_i^-)$.

Riemann Problem at a junction: Principle ...

- On all ingoing (resp. outgoing) roads i (resp. j), the initial datum U_i^- (resp. U_j^+) is known, and the actual outgoing (resp. ingoing) flux $q := q_i$ given by U_i^+(resp. q_j given by U_i^+) at the junction must satisfy the two Rankine-Hugoniot conditions (RH).
- **Full solution of the Riemann Pb** in the case of a **1-1 junction:** one ingoing (road 1) and one outgoing road (road 2), (e.g. asphalt-dirt road, or bottleneck ...): the maximal possible flux at junction is

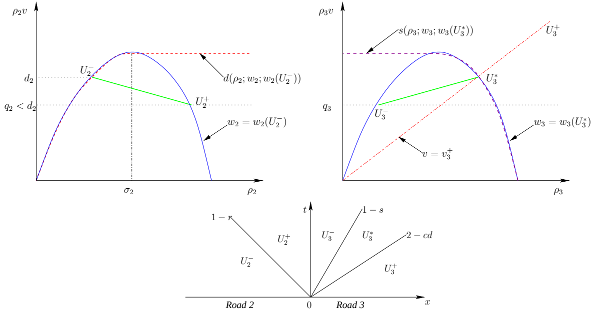
$$q = q_1 = q_2 = min(d(U_1^-), s(U_2^*)).$$

  Note that U_2^* is uniquely defined by the Riemann data U_2^+ and U_1^-.
- That defines two (in fact, a.e. a unique possible state on each road, e.g. $U_1^+ := U_1^-$ if $q = q_1^-$
- The solution is uniquely defined: $U_1^- \mid U_1^+ \mid\mid U_2^- \mid U_2^* ... U_2^+$. Moreover, in the particular case of a first order model (LWR), w is constant on each road and $U_2^* = U_2^+$. In this case, we retrieve the same results as e.g. Garavello-Piccoli.

1-1 Junction: Example

1-1 Junction: Example

One of the two waves below appears *either* on ingoing road 2 (left) or on outgoing road 3 (right). The actual flux is: $q = min(d(U_2^-), s(U_3^*))$, and on road 3, $w_3(U_3^*) = w_2(U_2^-)$!!, with here: $V_i(\rho) = V_{max} - p_i(\rho)$ and $w_i = v_i - V_i(\rho) + V_{max} = v_i + p_i(\rho)$.



2-1 junction: Homogenization

- Example : two incoming roads 1 and 2, with resp. black and white cars, with equal priority, and one outgoing road 3
- Then cars mix up on road 3, with average grey color. At the limit $\varepsilon \to 0$ we need a homogenized model (with conservation of the number of cars of each "color": (RH,ii)) ...

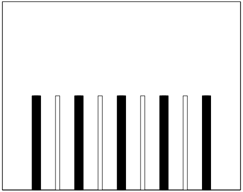


Figure 3: On an outgoing road ...

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Back to Homogenization

(with P. Bagnerini, sometimes here with different notations: $s := \tau = 1/\rho$ and $w = v - V(s)$ or $v - V(\tau)$. We consider a sequence of exact or approximate solutions, e.g. the Godunov approximation, possibly with one car per cell, to the Lagrangian system (3.5):

$$\begin{cases} \partial_t s - \partial_X v = 0\,,\ s := \rho^{-1} = \tau\,, \\ \partial_t w = 0\,,\ w = v - V(s) := v - V(\tau). \end{cases}$$

with initial data (or boundary conditions, at a junction) oscillating in w and τ (but not in v: that would be too dangerous! .. and these oscillations would be "killed" by the genuine nonlinearity). Typically, the mesh size in X and t is of order ε. We assume that:

$$v_0^\varepsilon \to v_0^*\,;\ w_0^\varepsilon \rightharpoonup w_0^* := w^*$$

In the corresponding solution, the velocity v^* is the strong limit of the non oscillating sequence (v^ε: v^* is the master unknown (BV) function ...

- But oscillations are preserved for $w^\varepsilon \equiv w_0^\varepsilon$ and for any function $F(v^\varepsilon, w^\varepsilon)$, e.g. for τ^ε
- For any F, weak limit (= "average") is described by Young measure : $< \nu_{X,t}, F(v,w) > := \int F(v,w) d\nu_{X,t}(v,w)$
- Since v^ε strongly converges, and since w is time-independent, the above integral equals $< \mu_X, F(v^*(X,t), w) > := \int F(v^*(X,t), w) d\mu_X(w)$
- ν and μ : probability measures, in v, w and in w respectively.

- The homogenized w is therefore: $w^* :=< \mu_X, w >$
- Since $V(\tau) := \tilde{V}(\tau^{-1})$ is strictly monotonous, $\tau = V^{-1}(v - w) = T(v, w) := T(v(X,t), w(X))$. Therefore, passing to the limit in the distribution sense in the Lagrangian system (3.5), we see that the homogenized τ, i.e.the weak-* limit of τ^ε:

$$\begin{cases} \tau^*(X,t) := T^*(X, v^*(X,t) := \\ < \mu_X,\ V^{-1}(v^*(X,t) - w) >=< \mu_X,\ T(v^*(X,t), w) > \\ = \int T(v^*(X,t), w)\, d\mu(w) := T^*(X, v^*(X,t)) \end{cases} \tag{4.1}$$

is a weak solution (and in fact, using classically Jensen's inequality in the averaging step of the Godunov scheme, is) an entropy weak solution to:

$$\partial_t T^*(X, v^*(X,t)) - \partial_X v^* = 0\ ... \tag{4.2}$$

- In (4.2), τ^* is naturally a function of v^* and X. We could invert again the roles of τ and v and write

$$\partial_t \tau^* - \partial_X F(X, \tau*) = 0, \tag{4.3}$$

  scalar conservation law whose flux is discontinuous in X, with $\partial F : \partial \tau > 0$: no resonance ...
- Integration in X of (4.3) leads to Hamilton-Jacobi equation : in periodic case, cf Lions-Papanicolaou-Varadhan.
- Here, by monotonicity, we deal directly with (4.2), and use the informations on Lagrangian system (L). In particular, an entropy η is convex in τ iff the associated flux $q \equiv q(v)$ is concave in v
- **Def:** v^* is an entropy solution to (4.2) if $\forall k$,

$$\partial_t |T^*(X, v^*(X,t)) - T^*(X,k)| - \partial_X |v^*(X,t) - k| \leq 0, \tag{4.4}$$

  which is equivalent by monotonicity to

$$\partial_t < \mu_X, |V^{-1}(v^*(X,t) - w) - V^{-1}(k - w)| > \\ -\partial_X |v^*(X,t) - k| \leq 0.$$

- **Theorem:** $\exists$ a unique entropy solution to (4.4)

2-1 Junction: Example

- Here, incoming roads 1 and 2 merge on outgoing road 3, with proportions α and $1-\alpha$.
- Typically, if e.g. $\alpha = 1/2$, then $\forall F(v,w)$, $< \mu_X,\ F(v,w) >= \frac{1}{2}\,(F(v,w_1^-) + F(v,w_2^-))$
- Again, since $V_3(\tau)$ is monotonous, we set: $T_3(v,w) := V_3^{-1}(v-w)$, and the homogenized τ: $\tau^*(X,t) := T_3^*(X, v^*(x,t)$ is given by:

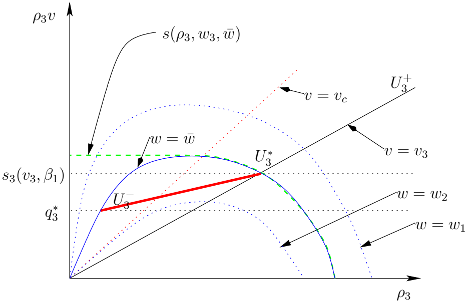
$$\begin{cases} \tau^*(X,t) =< \mu_X,\ T_3(v^*(X,t),w) > \\ := \frac{1}{2}\,(T_3(v^*(X,t),w_1) +\ T_3(v^*(X,t),w_2)) \\ = \frac{1}{2}\,(V_3^{-1}(v^*(X,t)-w_1) +\ V_3^{-1}(v^*(X,t)-w_2)) \end{cases} \tag{4.5}$$

- For any given v, the 1 and 2-drivers share the spacing, as in Figure 1.
- For any $v = v^*(X,t)$, the homogenized τ is **uniquely** defined by (4.5)
- If v varies, this relation defines a.e. in X a unique **homogenized fundamental diagram**, here in Lagrangian coordinates, associated with the average $w^* = (w_1+w_2)/2 =< \mu_X, w >$.

- These two relations describe the average conservation of space: **Attention: τ is additive, not ρ!** and the conservation of **the average** number of cars of each "color".
- The Eulerian counterpart is described in Figure below, from which one can construct the **homogenized supply (or demand) on road** 3.

Homogenized supply

The supply s on road 3 corresponds to the green/blue curve and to the unique point U_3^* on this curve with velocity v_3^+, with here $V^* = V_3^*, w = w^* := \bar{w}$



Note that the flux is maximal if aggressive drivers (coming here from road 1: bigger flux) take over...

Conclusion on junctions

- If there are several incoming roads and if w has an influence on the preferred velocity, then homogenization is needed on the ougoing roads
- Other ingredients are unchanged, e.g. here, in a $2-1$ junction with equal fluxes, compare d_1, d_2 (incoming) to (homogenized) outgoing $\frac{1}{2}\,s_3$
- In the general case, an additional criterion is needed: either impose ratios between ingoing fluxes or maximize total flux, and keep track of the destinations on outgoing roads. Note that the origin and destination can be viewew as additional components of w ...)
- The best way to maximize the total flux is to let the aggressive drivers take over ...
- The calculations of the homogenized problem are not that complicated, in fact are trivial if τ is initially defined wrt v, as in some discrete models, e.g. the Intelligent Driver Model ...

1. Discrete / Fluid Models
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    - The Eulerian System
    - Motivations. Lagrangian version
    - Link with microscopic models (FLM)
    - Lagrangian Godunov Scheme
    - Application: hybrid Lagrangian schemes
    - Passing to the limit(s)
3. Junctions
    - On a network
    - Ingoing Half-Riemann Problem
    - Outgoing Half Riemann Problem
    - Riemann Problem at a junction: Principle ...
    - 2-1 Junction: Homogenization
    - Homogenized Supply
    - Conclusion on junctions
4. With Relaxation. Traveling Waves and Oscillations
    - Remark: Whitham Subcharacteristic Condition
    - Smooth "simple waves" are generically Traveling Waves
    - J. Greenberg's periodic solutions for ARG. Extensions

Whitham Subcharacteristic Condition

- If we use the **same** hyperbolic scaling, relaxation term becomes $\frac{1}{\varepsilon}(V_e(\rho) - v)$, with $\varepsilon \to 0$: **zero-relaxation limit** problem.
- Whitham **Subcharacteristic Condition** is then necessary for stability: **(SC)** : on the equilibrium curve: $v = V_e(\rho)$, the characteristic speed of the formal equilibrium system, here

$$\partial_t \rho + \partial_x(\rho V_e(\rho)) = 0$$

must be **between** the two eigenvalues of the non-equilibrium system:

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \ q = \rho v \, , \\ \partial_t w + v \, \partial_x w = R(\rho, v) := \frac{1}{\varepsilon}(V_e(\rho) - v) \ : \end{cases} \tag{5.1}$$

i.e. $\Leftrightarrow 0 < V_e'(\rho) < V'(\rho)$: **"Convection must dominate relaxation". Pb: if so, our previous model is too stable (TVD), many others, e.g. Bando, too unstable ... Intermediate case?**

- *We assume e.g. that there is more than one small parameter, (e.g. two in the IDM,)* **and mostly a relaxation time small, but nonzero***. In addition, we assume that some weak form of (SC) is satisfied and prevents from crashes (Bando) or negative velocities (PW)* **(invariant regions)***.*
- *With a suitable* fixed *scaling, the RHS is a Lipschitz function of the solution. Therefore, the classical results (existence, uniqueness, continuous dependence in* L^1 *...) apply. Of course, we lose the TVD property.* Traveling wave solutions are "generic":

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- **Thm (Le Roux)** For a large class of systems, including (5.2) below , traveling waves are "generic" in the sense: **any smooth "simple wave"**, i.e. any smooth solution whose all components are functions of one of them, (e.g. of ρ) **must be a traveling wave.**

Smooth ”simple waves” are generically Traveling Waves

- Recall: simple waves are the ones which emerge in large time behavior
- Of course, discontinuous solutions (shocks or contacts) persist, since they can't ”see” the relaxation term. In contrast, in some sense, **”T-waves replace rarefaction waves when there is a RHS”.**
- **Proof of Thm:**

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \ q = \rho v \,, \\ \partial_t w + v \, \partial_x w = R(\rho, v) := (V_e(\rho) - v) \ : \end{cases} \tag{5.2}$$

Assume that v, q *and* $w = v - V(\rho)$ *are (unknown) functions of* ρ. *Then by (5.2,i), we have*

$$\partial_t \rho = -q'(\rho) \partial_{x\rho}. \tag{5.3}$$

Now divide (5.2,ii) by $R(\rho, v)$ *and use (5.3) to obtain, for some unknown function* F,

$$F'(\rho) \partial_x \rho = 1.$$

- *Therefore* $F(\rho(x,t)) = x - A(t)$. *Now, multiply (5.2,i) by* $F'(\rho)$, *so that, for some function* A

  $$F'(\rho)\partial_t\rho = -A'(t) = -F'(\rho).q'(\rho)\partial_x\rho = -q'(\rho).$$

  Differentiating this relation in x *or in* t *shows first that* $q''(\rho)\partial_x\,\rho \equiv 0$ *and then, using (5.3), that*
  $A''(t) = q''(\rho)\,\partial_t\rho = -q'(\rho)q''(\rho)\,\partial_x\rho \equiv 0.$
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- *Of course, this is only true locally ...*

- In the sequel, we will work in Lagrangian coordinates. We consider

  $$\begin{cases} \partial_t s - \partial_X v = 0, \\ \partial_t w = R(s,v) := V_e(s) - v\ , \end{cases} \qquad (5.4)$$

  assuming there are given reasonable functions such that the relations

  $$v = V(s,w) \Leftrightarrow w = W(s,v) \Leftrightarrow s = S(v,w)$$

  are equivalent and that, e.g. for $w = v - V(s)$, we have:

  $$v = V(s) + w = V(s) + W_e(s) = V_e(s)) \Leftrightarrow w = W_e(s) = V_e(s) - V(s)$$

J. Greenberg's periodic solutions for ARG. Extensions

- Here, we show how to construct periodic solutions of (5.4) with one T-wave $U_- U_+$ and one adjacent shock $U_+ U_-$.
- First, we seek a T-wave $U(\xi) := U(X + ct)$, $c > 0$ (U travels backwards) in Lagrangian coordinates, with e.g. $w = v - V(s)$

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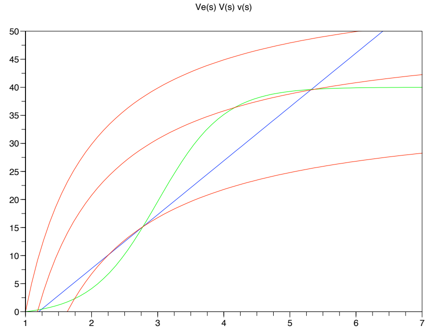
- Subcharacteristic Condition: $-V'(s) < -V_e'(s) < 0$ **only** satisfied on eq. curve for small or large s: invariant regions $v \geq 0$, no crash ...
- A T-wave connecting U_- to U_+ must satisfy: $c\dot{s} - \dot{v} = 0$ and (therefore) on the straight line $U_- U_+$, we must have:

$$\dot{s}(\xi) = \frac{R(s, v = cs + C)}{c(c - V'(s)} := \frac{N}{D}:$$ N and D must vanish together, i.e.

D must vanish at the intersection point $U_0 := U_- U_+ \cap \{v = V_e(s)\}$: since $D = c\frac{d}{ds}(w(s, v = cs + C))$, the level curve $\{w(U) = w(U_0)\}$ must be **tangent** at U_0 to the straight line $U_- U_+$.


Ve(s) V(s) v(s)
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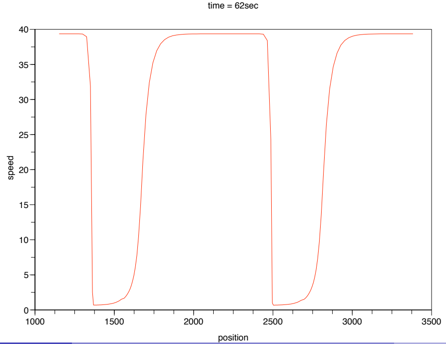
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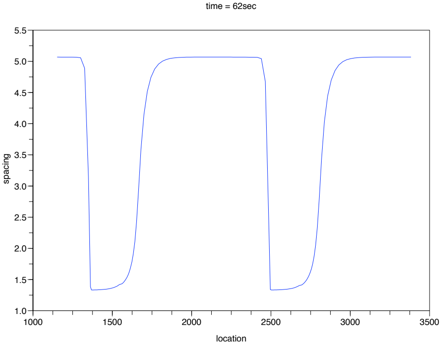
position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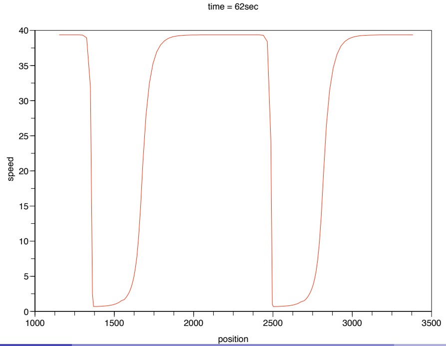
location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speed
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3500
position

- **Existence** of such a solution (heteroclinic orbit) by intermediate value Theorem, when $U_\pm \in$ "(SC) stable" region of $\{v = V_e(\tau)\}$. Figure. Uniqueness??
- Similar solutions exist with nearby endpoints $U_\pm$ not at rest (thus reached in finite time), with $w(U_-) = w(U_+)$. Then the T-wave can be interrupted (before reaching equilibrium) by an adjacent shock wave U_+U_- with **same** speed (Rankine-Hugoniot): $\exists$ periodic solutions, typically on a ring road
- Stability of such waves? How relevant is linear stability analysis ??

An example: the Intelligent Driver Model

With some modifications (e.g. on the length l) this model writes

$$\begin{cases} \dot{x}_j = v_j, \\ \dot{v}_j = a\,[1 - (\frac{v_j}{v_0})^m - (\frac{s_b(v_j) - v_j \frac{v_{j+1}-v_j}{2ab}}{x_{j+1} - x_j - l})^p], \end{cases} \tag{5.5}$$

where

$$s_b(v) := s_0 + s_1 \frac{v}{v_0} + Tv,$$

(note: desired spacing as a function of v, not the converse!), with $m = 1, 2$ or 4, $p = 1$ or 2, and

$$a = b = 1m/s^2;\ v_0 = 33m/s:\ l = 5m;\ s_0 = 1m;\ s_1 = 10m;\ T = 1sec.$$

We introduce reference quantities: x_r, t_r, v_r. Assume that $v_r = x_r/t_r = v_0$. Two (small) dimensionless parameters appear $\varepsilon := \frac{l}{x_r}$, $\mu := \frac{a\,l}{v_0^2}$.
Typically, we choose: $x_r = 1000m$; $t_r = 30sec$, $v_r = v_0 = 33m/sec$ ...

so that $\varepsilon := \frac{5}{1000} = \frac{1}{200} = \mu := \frac{5}{33^2}$.
The term: $\frac{x_{j+1}-x_j-l}{x_r}$ becomes: $\frac{x_{j+1}-x_j}{\varepsilon} - 1 := s_j$ in rescaled coordinates, and the system rewrites

$$\begin{cases} \dot s_j = \frac{v_{j+1}-v_j}{\varepsilon}, \\ \dot v_j = \frac{\mu}{\varepsilon}\,[1 - v_j^m - (\frac{s_b(v_j) - v_j\,\frac{\varepsilon}{\mu}\,\frac{v_{j+1}-v_j}{\varepsilon}}{s_j})^p], \end{cases} \tag{5.6}$$

with now $s_b(v) := a_0 + a_1 v + a_2 v$ and $a_0 = \frac{s_0}{l}$, $a_1 = \frac{s_1}{l}$, $a_2 = \frac{Tv_0}{l}$.
Now, first $\mu = \varepsilon$ and next, say for $p = 1$, (5.6,ii) rewrites:

$$\dot v_j = [A(v) - s_b(v_j)/s_j] + (v_j/s_j)\,\dot s_j, \text{ with } A(v) := 1 - v_j^m. \tag{5.7}$$

Now, multiply both members by $1/v_j$ **(integrating factor)**, and define $w = W(s,v) := ln(s/v)$. Note that: $\frac{1}{v}\dot v - \frac{1}{s}\dot s = \frac{\partial W}{\partial v}\dot v + \frac{\partial W}{\partial s}\dot s = \dot w$. The above equation rewrites:

$$\dot w = \frac{1}{v}\,[A(v) - \frac{s_b(v)}{s}] = \frac{A(v)}{v}\,[1 - \frac{s_e(v)}{s}], \tag{5.8}$$

with $s_e(v) := \frac{s_b(v)}{A(v)}$. Finally, we obtain:

$$\begin{cases} \dot{s}_j = \frac{v_{j+1} - v_j}{\varepsilon}, \\ \dot{w}_j = \frac{A(v_j)}{v_j}\,[1 - \frac{s_e(v_j)}{s_j}]\,; w = ln(s/v), \end{cases}$$

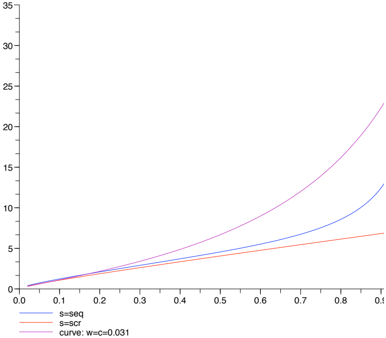
whose natural macroscopic version is thus (cf (5.4):

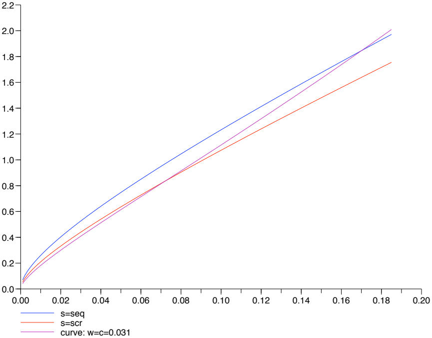
$$\begin{cases} \partial_t s = \partial_X v, \\ \partial_t w = \frac{A(v)}{v}\,[1 - \frac{s_e(v)}{s}]\,; w = ln(s/v), \end{cases}$$

for which e.g. $\{0 \le v \le v_{max}; 0 < w_{min} \le w = s/v\}$ is **invariant**.

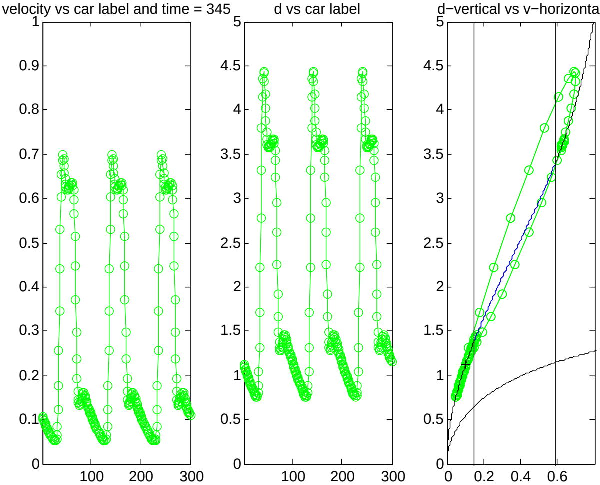
- **For $p = 1$, the natural macroscopic version of IDM is a particular case of ARG model.**
- **For$p = 2$, idem if we neglect the term quadratic in $(v_{j+1} - v_j)$, which is legitimate if $\mu << \varepsilon$.**
- Natural question: can we exhibit Traveling Waves ARG style for this system? Answer: NO for this one precisely, for hidden geometric reasons. Good hope for variants of this system, see below.

Example of difficulty: analyzing the nature and the number of intersection points between these curves (or their tangents) is not easy ... here for the case $p = 2$: the (non ?)-existence of T-waves previous style dramatically depends on these details ... which can be modified with still very reasonable qualitative properties ... e.g. in (5.7), we had: $\frac{v}{s} = -\frac{\partial W}{\partial v} / \frac{\partial W}{\partial s}$, which determined w ... Work in progress. Collaborations ...





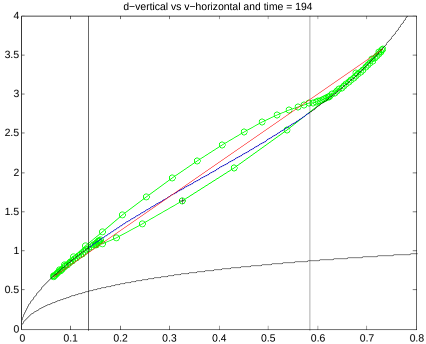
Examples of numerical results for a variant: J.Greenberg. Here, the waves look like for Bando's model: periodic orbits, no shock, at the expenses of adding a diffusion term *in the first equation!!*: philosophy of "equivalent equation to higher order"... see below

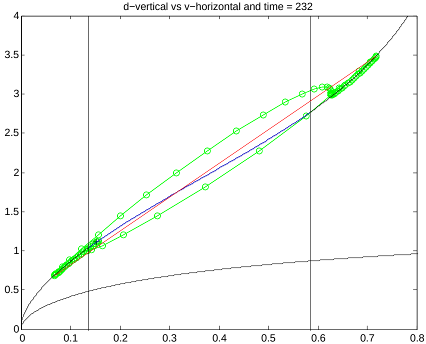


Indeed, here, the macroscopic equation is, say with $l = 1$:

$$\begin{cases} \partial_t s = \partial_X v + (1/2)\, \partial^2_{XX} v, \\ \partial_t w = \frac{A(v)}{v} \left[1 - \frac{s_e(v)}{s}\right];\, w = ln(s/v). \end{cases}$$

It admits T-wave solutions: either closed periodic heteroclinic orbits connecting the two saddle points $M_i = (v_i, s_e(v_i))$, $i = 1, 3$. The crucial equilibrium point M_2 is a center. Here and below, transitory regime ...





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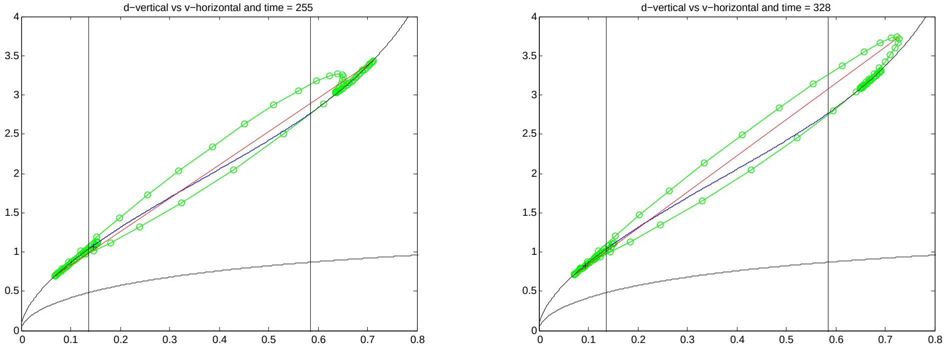
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d−vertical vs v−horizontal and time = 328
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1.5
2
2.5
3
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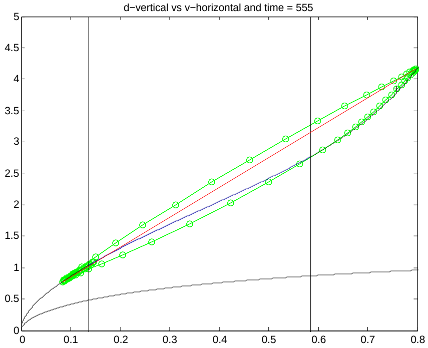
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

...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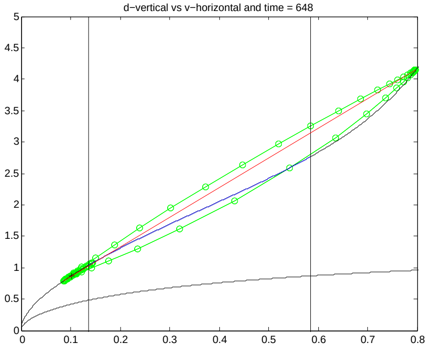
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0.4
0.5
0.6
0.7
0.8

Additional Remarks. Conclusion

- The original AR model is too stable for describing realistic oscillations
- However, we need its convective short time stability properties for avoiding bad things ...
- Adding a relaxation term which violates the subcharacteristic condition, but (**only** at **intermediate** densities), can give nice qualitative results, still avoiding any crash or negative speed (invariant regions), as in the ARG type of models (crucial role of Jim!). Compare with Bando or PW ...
- Same approach can be applied e.g. to (truncated) IDM, with a neat priority to discrete models (dispersion relation ...) away from $\{\rho = \rho_{max} = 1, v = 0\}$. Many possible (neater) modifications near this dangerous region ...
- Like in many asymptotic expansions, adding a higher order (diffusive or dispersive) term in one of the PDEs can be very useful for getting oscillating solutions for the discrete system, even though, rigorously, the PDE is wrong! Modified equation philosophy: priority to discrete models ...

A few comments and references

Among many other aspects ...

- Aw-Rascle (AR), SIAP 2000: "Resurrection": initial model, Riemann Problem.
- Aw-Klar-Materne, SIAP 2002: Lagrangian view, rigorous derivation from microscopic models. See this paper for details on the convergence of Godunov scheme, either to the Follow the Leader ODE Model when $\Delta t \to 0$ or directly to the Lagrangian equivalent system (L) when both ΔX and $\Delta t \to 0$ (with a fixed ratio, under the CFL condition) and the related commutation of limits.
- Much more recently, including with a relaxation term and/or in the case of vacuum, see Godvik and Hanche Olsen, 2008, see also Colombo-Marcellini-Rascle, SIAP 2010, using an interpretation of a Phase Transition Model of Traffic of Colombo as a variant of AR model already observed by Lebacque *et al*, who gave also a (slightly more general, but much nicer) presentation of (AR) as GSOM (Generalized Second Order Models).

- See also related works in Berthelin-Degond *et al*, ARMA 2008, M3AS 2008, with cartoons of traffic jams described as formal asymptotic limits of (AR), leading to sticky (in)compressible clusters, and see later extensions by Herty *et al* for applications to junctions.
- For the (AR) system with relaxation, see J. Greenberg, SIAP 2001, SIAP 2004 and 2007, see also the (unpublished) PhD Thesis of Aw, and Greenberg-Klar-Rascle, SIAP 2003. See also a paper Mauser-Moutari-Siebel: relaxed model with sometimes negative time relaxation time.
- For a study of a hybrid scheme (discrete near junctions, continuous elsewhere, with Lagrangian interfaces, see Moutari-Rascle, SIAP 2008.
- For relations between (AR) and kinetic models, see Klar-Wegener, SIAP, early 2000', and more recently papers by Herty, Illner and co-workers, since 2008.

About junctions and Homogenization

- For the application to networks or more precisely to the Riemann Problem at junctions, using the notions of demand an supply (Lebacque)
- For the study of the homogenized Lagrangian model and the proof of existence and uniqueness of its solution, see Bagnerini-Rascle, SIMA 2003
- For the application to junctions, strongly based on the previous paper, see Herty-Rascle, SIMA 2006, Herty-Moutari-Rascle, NHM 2006, and related papers by the same Authors
- For many networks problems based on first order models (LWR) and a neat discussion of additional criteria needed at junctions, we refer e.g. to the book of Garavello-Piccoli.